General Relativity | Week 7

Special Relativity: Formulated by Einstein in 1905

Geometric reformulation: Minkowski 1908

Space and time: $(\mathbb{R}^{3+1}, \eta)$

Inertial observers: Move in <u>straight</u> <u>timelike</u> lines

Let's restrict to $\mathbb{R}^{1+1}$ for simplicity. In the <u>orthonormal</u> frame of reference $\{e_0, e_1\}$ of an inertial observer γ:

If γ' is a different inertial observer:
They have a different associated orthonormal frame $\{e_0', e_1'\}$

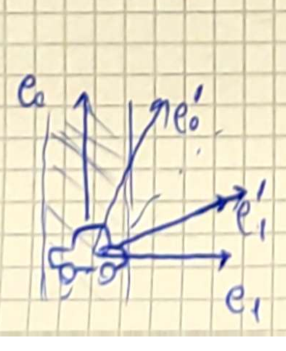
Note: $\{e_0, e_1\}$ is related with $\{e_0', e_1'\}$ through a <u>linear isometry</u> of $(\mathbb{R}^{1+1}, \eta)$.

Note: light moves at the same speed for both observers.

- The set of "constant time" for γ: Lines orthogonal to e_0 (so parallel to e_1)

  $\rightsquigarrow$ γ and γ' have different notions of "simultaneous events"

- Length of an object: Distance of endpoints as measured at a constant time slice $\Rightarrow$ depends on the observer!

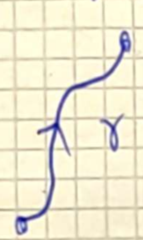


The length of the car seems to "shrink" for observers moving relative to the car!

Principle of special covariance: Physical laws should have the same form for all inertial observers (so: They have to be invariant under linear isometries).

So: Physically relevant quantities: Geometric in nature.

E.g: Proper time: The time felt by an observer (moving along a curve γ) : Is the Lorentzian length of the curve



• Twin paradox: ...

Overall: Special relativity is the study of the geometry of $(\mathbb{R}^{1+3}, \eta)$

inertial observers $\Longleftrightarrow$ timelike geodesics & orthonormal frames.

General relativity: Incorporation of gravity into special relativity

Einstein realized that it had to be based on two principles:

1. Principle of equivalence: Free-falling observers experience the same physical laws as inertial ones.

2. Principle of general covariance:
   Physical laws must be "invariant" under changes of "free-falling frames".

Unifying set-up:

Spacetime is a Lorentzian manifold (M^{3+1}, g)

g: Incorporates the properties of gravity

- Local frame $\leftrightarrow$ local coordinate system
- Free-falling observers: Move along <u>timelike geodesics</u>.

①: In normal coordinates (in which the geodesic becomes a straight line): $g \approx \eta$ to top 2 orders

②: Translates to "Physical laws should be of tensorial (aka geometric nature).

How does matter tell the spacetime how to curve?

In <u>Newtonian gravity</u>: Free-falling observer: $t \longrightarrow (x^1(t), x^2(t), x^3(t))$,

$$\begin{cases} \ddot{x} = -\nabla \Phi & \leftarrow \text{gravitational potential} \\ \Delta_{\mathbb{R}^3} \Phi = 4\pi \rho & \leftarrow \text{mass-density} \end{cases}$$

↑ Poisson's equation.

In general relativity:

The 1st equation becomes the geodesic equation:

Free-falling observer: $s \longrightarrow (x^0(s), x^1(s), x^2(s), x^3(s))$,

$$\ddot{x}^a = -\Gamma^a_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma$$

Since $\Gamma \sim \partial g \longrightarrow \Phi$ should correspond to g.

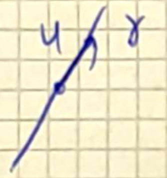
What would be the analogue of Poisson's equation?

In special relativity: Mass / energy / momentum are <u>not</u> separately invariant under changes of frame.

"Combined" object: Energy momentum tensor $T_{\mu\nu}$

- Comes from the field-theoretic description of the matter field: Given a Lagrangian, there is an "algorithm" to compute $T_{\mu\nu}$

- For a point particle of mass m moving along a timelike curve γ: (with unit tangent vector u):



$$T_{\mu\nu} = m \cdot u_\mu u_\nu \cdot \delta_\gamma \leftarrow \text{delta function}$$

- If e_0 is timelike and unit: $T_{00} = T(e_0, e_0)$ is the energy density of matter as measured by an observer moving in the direction of e_0.

- Conservation of energy: $\nabla^a T_{a\beta} = 0$.

So: The correct equation:
- 2^{nd} order in g
- geometric

} It should involve $R_{a\beta\gamma\delta}$, $g_{a\beta}$

- Must have $T_{\mu\nu}$ on the right hand side.

Necessarily: $\boxed{Ric_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi\, T_{\mu\nu}}$ $G = c = 1$

- The only divergence-free contraction of the Riemann tensor

Λ: cosmological constant, has units $1/[L]^2$.

Originally: $\Lambda = 0$

(today: $\Lambda > 0$ at cosmological scales)

Why is 8π the correct constant on the right hand side?

So that we recover Poisson's equation in the post-Newtonian limit!

That is to say:
- Weak gravitational field: $g_{\mu\nu} = \eta_{\mu\nu} + \hat{g}_{\mu\nu}$, $\hat{g} = O(\varepsilon)$
- Slow motion (observer & matter)
- Free-falling observer $x^a(\tau)$ with
  $$\dot{x}^0(\tau) = 1 + O(\varepsilon^1), \quad \dot{x}^i(\tau) = O(\varepsilon^1), \; i = 1,2,3$$
- $T_{ij}, T_{0i} \ll T_{00}$ (so that $T_{00} = \rho + \text{l.o.t.}$)
- Nearly static gravitational field
  $$\left(\partial_0 \hat{g}_{\alpha\beta} \approx 0 \right).$$

Then: Geodesic equation:

$$\left(\frac{d}{d\tau} = \frac{d}{dx^0} + \text{l.o.t.}\right): \quad \ddot{x}^i(\tau) = -\Gamma^i_{00} + \dots = +\frac{1}{2} \partial_i \hat{g}_{00} + \dots$$

So in the "limit": The gravitational potential must satisfy (up to an additive constant):

$$\boxed{\Phi = -\frac{1}{2} g_{00}}$$

Moreover, from ② $Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$, if I take the trace:

$$R - \frac{4}{2} R = 8\pi \, tr_g T \overset{①}{\Longrightarrow} Ric_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} tr_g T \cdot g_{\mu\nu} \right)$$

$\overset{\mu,\nu=0}{\Longrightarrow}$ $$\mathrm{Ric}_{00} = 8\pi\left(T_{00} - \tfrac{1}{2}\,\mathrm{tr}T\cdot g_{00}\right)$$
$$= 4\pi\, T_{00} + \dots$$
$$= 4\pi\rho$$

And: $\mathrm{Ric}_{00} = \sum_{i=1}^{3} R_{i0i0}$

Since $R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\alpha\gamma} g_{\beta\delta} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\beta\delta} g_{\alpha\gamma}\right) + O((\partial g)^2)$

and $\partial_0 g_{\alpha\beta} \approx 0$:

$$R_{i0i0} \approx -\frac{1}{2}\partial^2_{ii} g_{00} + \dots$$

$$\Rightarrow \mathrm{Ric}_{00} = -\frac{1}{2}\sum_{i=1}^{3}\partial^2_{ii} g_{00} = \Delta_{\mathbb{R}^3}\Phi + \dots \qquad \boxed{}$$

We will consider the Einstein equations in any dimension.

· Vacuum equations: $\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0$, (M^{n+1}, g)

If $n \geq 2$: $\Leftrightarrow \mathrm{Ric}_{\mu\nu} = 0$ (In 1+1 dim: All Lorentzian manifolds satisfy $\mathrm{Ric} - \frac{1}{2}R\cdot g = 0$)

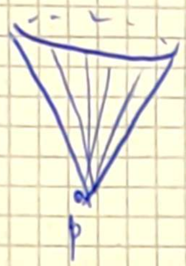
Unlike Newtonian theory: There are non-trivial solutions in vacuum!

- Earlier attempt at a geometric theory of gravity: Nordström.

<u>Simplest solution</u> of $\mathrm{Ric} = 0$: Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$

("The trivial solution")

Causal structure: $\forall p \in (\mathbb{R}^{n+1}, \eta)$, $J^+(p)$ is the same as $J_p^+ \subseteq T_p\mathbb{R}^{n+1}$ if we identify $T_p\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$



$\partial J^+ = J^+ / I^+$

It is globally hyperbolic.

We would like to understand a bit better the global causal structure of $(\mathbb{R}^{n+1}, \eta)$ asymptotically at "infinity".

<u>Note:</u> If $g, \tilde{g}$ are Lorentzian metrics on M with $g = \Omega \cdot \tilde{g}$, $\Omega > 0$, they have the <u>same causal structure</u> (e.g. $I^{\pm}(p)$, Cauchy hypersurfaces etc)

So we can try to <u>conformally compactify</u> spacetimes, in order to "read off" the causal structure more easily.

<u>1+1 Minkowski:</u> $\eta = -dt^2 + dx^2$

Double null coordinates: $u = t - x$, $v = t + x$

$\eta = -du\,dv$

↑ Coordinate vector fields are null, so they "capture" better the geometry

Compactify: $(u,v) \to (\tilde{u}, \tilde{v})$

$$\mathbb{R}^{1+1} \longrightarrow D \subseteq \mathbb{R}^{1+1}$$

$$\tilde{u} = \tilde{u}(u) = \operatorname{Arctan}(u)$$
$$\tilde{v} = \tilde{v}(v) = \operatorname{Arctan}(v)$$

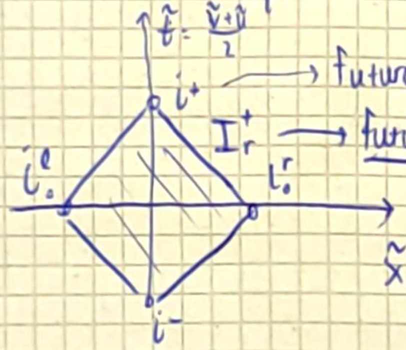
(any choice of functions with finite ranges would do)

Then: $(\tilde{u},\tilde{v}) \in (-\frac{\pi}{2},\frac{\pi}{2}) \times (-\frac{\pi}{2},\frac{\pi}{2})$

$$du = \frac{1}{\cos^2\tilde{u}}\, d\tilde{u}, \qquad dv = \frac{1}{\cos^2\tilde{v}}\, d\tilde{v}$$

$$\Rightarrow \eta = -\frac{1}{\cos^2\tilde{u}\cdot\cos^2\tilde{v}}\, d\tilde{u}\, d\tilde{v} = \frac{1}{\cos^2\tilde{u}\cdot\cos^2\tilde{v}}\, \tilde{\eta},$$

Where $\tilde{\eta} = -d\tilde{u}\,d\tilde{v}$: Minkowski metric in the $(\tilde{u},\tilde{v})$ plane



$i^+ \to$ future timelike infinity

$I_r^+ \to$ future right null infinity: endpoints at infinity in the original spacetime of right-moving future directed null geodesics

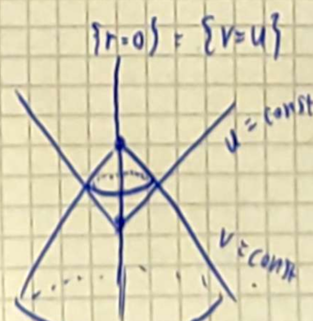
If I choose a different conformal compactification: The image would still be a rectangle with null sides.
So the causal structure of the boundary at infinity is independent of those choices.

So what we did above:
We constructed a conformal map $F: (\mathbb{R}^{1+1}, \eta) \to (N, \tilde{g}) = (\mathbb{R}^{1+1}, \tilde{\eta})$

with pre-compact image.

3+1 Minkowski:

$$\eta = -dt^2 + dr^2 + r^2 g_{S^2}$$



$u = t-r$, $v = t+r$
$(r \geq 0 \Rightarrow v \geq u)$:

$$\eta = -du\,dv + \frac{1}{4}(v-u)^2 g_{S^2}$$

As before: $\tilde{u} = \mathrm{Arctan}(u)$, $\tilde{v} = \mathrm{Arctan}(v)$,

So $\underset{\in S^2}{(u, v, \omega)} \longrightarrow (\tilde{u}, \tilde{v}, \omega)$

$$\eta = \frac{1}{\cos^2\tilde{u}\cdot\cos^2\tilde{v}} \underbrace{\left(- d\tilde{u}\, d\tilde{v} + \frac{1}{4}\sin^2(\tilde{v}-\tilde{u})\, g_{S^2}\right)}_{\tilde{g}}$$

With $\tilde{v} \geq \tilde{u}$, $\tilde{v}, \tilde{u} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

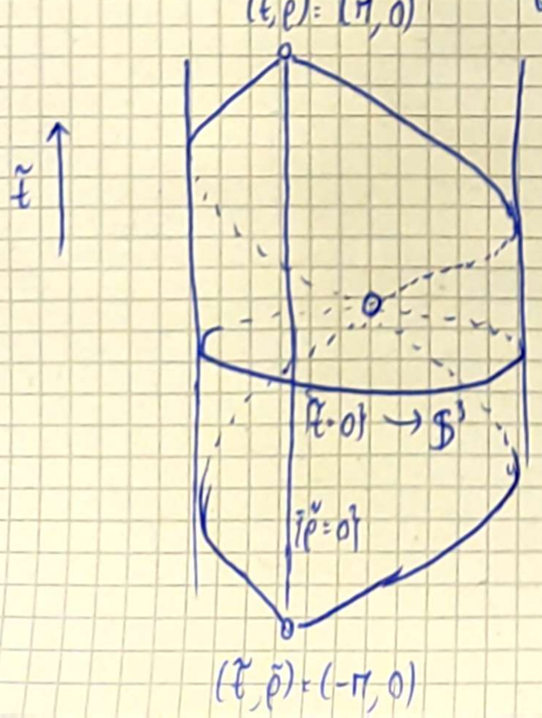
Is $\tilde{g}$ a familiar metric?

If $\tilde{t} = \tilde{v} + \tilde{u}$, $\tilde{\rho} = \tilde{v} - \tilde{u}$: $(0 \leq \tilde{\rho} < \pi)$:

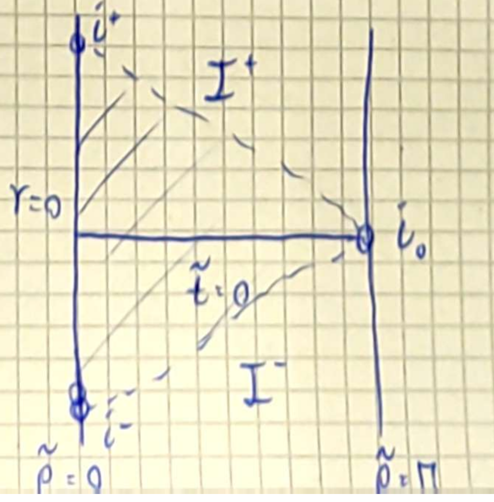
$$4\tilde{g} = -d\tilde{t}^2 + \underbrace{d\tilde{\rho}^2 + \sin^2\tilde{\rho}\, g_{S^2}}_{g_{S^3} \text{ in polar coordinates}}$$

So the map we constructed maps $(\mathbb{R}^{3+1}, \eta)$ conformally into the domain of $(\mathbb{R} \times S^3, -d\tilde{t}^2 + g_{S^3})$ characterized by

$$0 \leq \tilde{\rho} < \pi, \quad |\tilde{t}| < \pi - \tilde{\rho}$$



Or, if I "forget" the S^2 direction:



I^+ is where far away radiation ends up

In the above, the picture on the right: Penrose diagram of a spherically symmetric spacetime

(every point corresponds to an $\mathbb{S}^2$ sphere of rotation in the original spacetime, except at $\theta = 0, \pi$, i.e. at the axes of rotation)

In general, to construct the Penrose diagram of a surface-symmetric spacetime: Conformally compactify & quotient out isometries.

For $(\mathbb{R}^{3+1}, \eta)$: A point at $\mathcal{I}^+$ is the point where parallel null straight lines end up.